Title: Exploring Hyperspace with the Geometric Product
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Abstract: Overviews on geometry in 4-space for a general audience are enhanced when geometric products for figures and boundaries of figures are included. In particular, the classic work by Manning (1914), "Geometry of Four Dimensions", benefits when many of the figures it treats are also derived using geometric products. Geometric products lead to other high-dimensional figures and create new avenues for research.

1. Introduction

The geometric product provides a direct and intuitive route to higher-dimensional structures and figures. It is used within the province of geometric spaces much as the Cartesian product is used in the province of sets. One goal in this exploration is to use the geometric product to help define and dissect four-dimensional figures composed of tori and "wormholes" as examined in Section 3. Geometric products applied to boundaries as presented here might have averted conflicting definitions for toroidal objects in Manning (1914) and Coxeter (1974). Also, this investigation of geometric shapes of products through planar cross sections or bundles of factors, being outside the scope of topological studies, has the potential to open areas for further research.

Notations used for geometric factors or products have superscripts that give their intrinsic dimension. The words "surface" or "face" for two-dimensions, "solid" for three-dimensions, and "hypersolid" for four-dimensions also select an intrinsic dimension. Factors with commonly used symbols include an interval or line segment (I or B^1), circle (S^1), disk (D^2 or B^2), sphere (S^2), and solid ball (B^3). The flat torus product $S^1 \times S^1$ has the symbol T^2. For low-dimensional examples with two factors, the product of two intervals makes a square surface (I x I), an interval times a circle yields a cylindrical surface, and an interval times a disk makes a solid cylinder. Each factor is embedded in its own coordinate space (Euclidean R^s and R^t); thus, for two polygons, one will reside in the *xy*-plane and the other in the *wz*-plane. These planes intersect only at the origin as is only possible in 4-space or higher dimensional space and their metric is assumed to be the standard Euclidean metric. A product of a geometric factor in R^s and one in R^t lies in the product space R^{s+t}. A polygon (P_n^1) is defined as a ring of *n* line segments joined at their endpoints, and P_n^2 is the surface bounded by P_n^1. Thus, I x I can also be written P_4^2. This paper will use regular polygons and avoids *d*-dimensional intervals for $d>1$. The boundary of an object with a boundary is indicated by the boundary operator "∂". It is used, for example, to denote P_n^1 as ∂P_n^2 or the boundary of I x I as ∂P_4^2.

2. Products of Intervals and Polygons

Factors in this exploration will progress from intervals and polygons to spheres and disks. We begin by examining factors and products in R^2 and R^3 that yield the familiar regular orthotopes - the square and the cube. As shown by these examples, the boundary of a geometric product of two "factors with a boundary" is composed of the disjoint interiors of two orthogonal products and the intersection of these products (Kelley, 1955, p. 103). This intersection is the common boundary enclosing these two orthogonal products within their intrinsic dimension. One product is composed of the first factor times the boundary of the second factor and the other the second factor times the boundary of the first factor. Their common boundary is the geometric product of the boundaries of the two factors.

In our first example, the square polygon (∂P_4^2) is composed of disjoint interiors of congruent orthogonal products $\partial I \times I$ and $I \times \partial I$ with their common boundary the four vertex points (Figure 1 left). These parallel line segments are joined at their endpoints by the boundary product $\partial I \times \partial I$: $\partial(\partial I \times I) = \partial(I \times \partial I) = \partial I \times \partial I$ (Figure 1 right).

Figure 1. Components of a square.

Next, we find related expressions for two products in 3-space that yield a cube $\partial(I \times I \times I)$. First, sides are constructed: $\partial(I \times I) \times I = P_4^1 \times I$ (Figure 2 left). This open square prism $P_4^1 \times I$ is then joined to the interiors of two square surfaces $(P_4^2 \times \partial I) - \partial(P_4^2 \times \partial I)$ to construct $\partial(I \times I \times I)$ (Figure 2 right). Parallel squares $\partial P_4^2 \times \partial I$ become a two-part common boundary: $\partial(P_4^1 \times I) = \partial(P_4^2 \times \partial I) = P_4^1 \times \partial I$. Note that one product $(P_4^1 \times I)$ that comprises the cube is a ring of four square surfaces. Analogous rings of orthotopes constitute the hypercube.

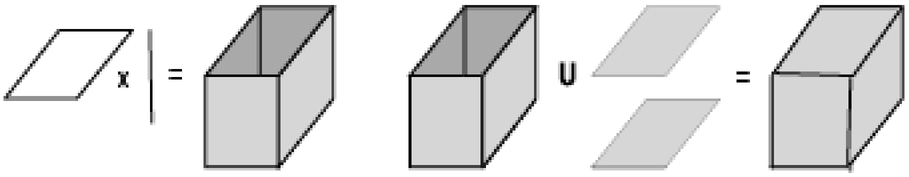


Figure 2. Products that constitute a cube.

The hypercube, a well-known orthotope in R^4, is in fact the boundary of the hypersolid hypercube $P_4^2 \times P_4^2$. This solid figure, $\partial(P_4^2 \times P_4^2)$, is composed of disjoint interiors of two congruent, orthogonal products $P_4^1 \times P_4^2$ and $P_4^2 \times P_4^1$ within a single common boundary. These orthogonal products are connected objects and take the form of rings of four solid cubes. An adjacent pair of cubes can be made to lie in the same 3-dimensional subspace by a rotation in R^4 that leaves their common face fixed. If each of the solid rings comprising a hypercube is cut where two solid cubes meet and rotated into a common 3-space, a fold-out decomposition is the result (Figure 3). The common boundary between and enclosing the rings of four solid cubes in a hypercube is a "polyhedral torus" that can be cut along a cycle of edges (a meridian) and an orthogonal cycle of edges (a longitude) to allow it to unfold from its embedding in R^4 and reside in R^2 (Banchoff, 1990). The result is a flat 4 x 4 rectangular grid as a fold-out decomposition. Each solid ring has this boundary: $\partial(P_4^1 \times P_4^2) = \partial(P_4^2 \times P_4^1) = P_4^1 \times P_4^1$. As with $\partial I \times \partial I$ in the square, $P_4^1 \times P_4^1$ has congruent factors.

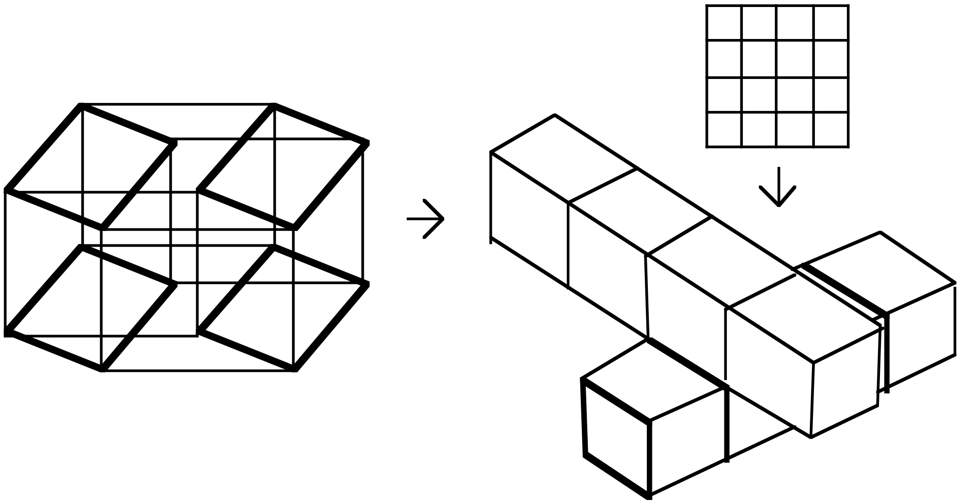
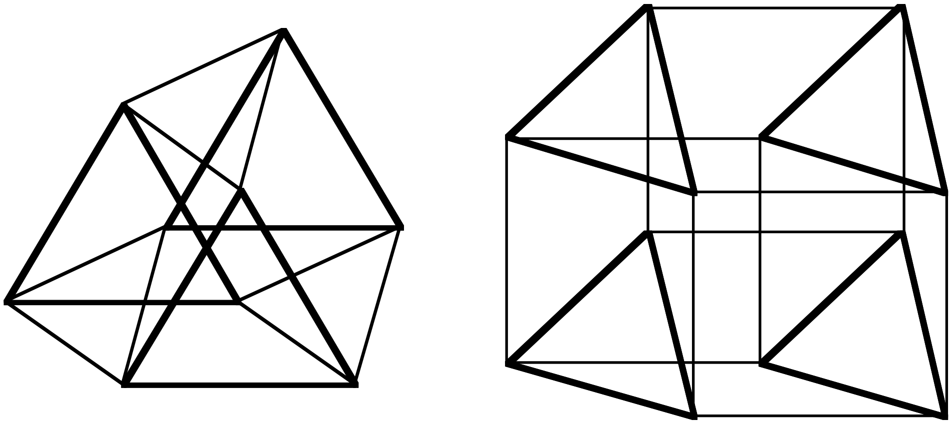


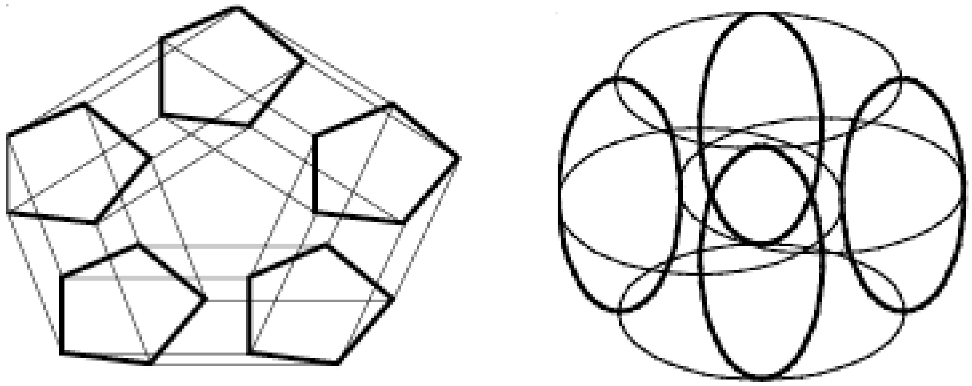
Figure 3. Fold-out of a projection of a hypercube with the corresponding polyhedral torus unfolded in the upper right.

The solid boundary of $P_n^2 \text{ x } P_m^2$ has been referred to as a double prism (Manning, 1914). The hypercube $\partial(P_4^2 \text{ x } P_4^2)$ was our first example in this series of figures consisting of two rings of solid polyhedra. Double prisms have also been defined as four-dimensional objects that are composed of products of polygonal faces (Coxeter, 1974). Coxeter's product of faces is the hypersolid product $P^2 \text{ x } P^2$ instead of the solid rings (solid tori) comprising the double prism $\partial(P_n^2 \text{ x } P_m^2)$ defined by Manning. Outlines of orthogonal projections of three double prisms are shown in Figure 4. Other than an orthogonal projection such as the one in Figure 4, $\partial(P_3^2 \text{ x } P_3^2)$ has been illustrated as a central "Schlegel" projection into 3-space (Sommerville, 1958, p. 108).



Triangular double prism

Square triangular double prism



Pentagonal double prism

Flat torus

Figure 4. Outlines of projections of four-dimensional figures composed of two tori in total contact with each other. Three-dimensional models of double prisms (made with a glue gun and toothpicks) are helpful.

Remarkably, each double prism is enclosed by only a single polyhedral torus common boundary $P_n^1 \text{ x } P_m^1$. For the triangular double prism this is a product of two triangles $P_3^1 \text{ x } P_3^1$ whose fold-out is a three-by-three rectangular grid. Different polygons such as the square and triangle in Figure 4 also yield a product that is a common boundary of a double prism. Thus, the polyhedral torus common boundary for $\partial(P_3^2 \text{ x } P_5^2)$ will be a closed 3 x 5 rectangular grid. For $P_n^1 \text{ x } P_m^1$ we have a "polygon of polygons" that embeds in R^4. Stacks of figures such as the subsets of polygons in this product are called bundles (Weeks, 1985, p. 231).

3. Products of Circles and Spheres

Polyhedral tori are angular versions of an important closed surface ($S^1 \times S^1$ or T^2) that isometrically embeds in R^4. This Euclidean figure is a flat torus that consists of bundles of longitudinal circles from one S^1 factor and meridian circles from the other S^1 factor (Figure 4). It differs geometrically but not topologically from the ordinary "doughnut" torus in 3-space. In fact, T^2 cannot be isometrically embedded in 3-space: T^2 has the same circumference for all its longitudinal circles. Projecting T^2 into R^3 creates a figure that has self-intersections. These geometric distortions in 3-space result from confinement in a space of fewer dimensions than are required ($s + t$ in Section 1) for isometrically embedding this product. Our various flat tori are recovered from fold-out rectangles by rejoining (identifying) opposite edges in R^4.

A wormhole is usually defined as an $I \times S^2$ tunnel joining two R^3 universes. In addition, $S^1 \times S^2$ will be regarded as a closed wormhole. As was done in Section 2, examining analogous models in R^2 and R^3 helps visualizing these wormhole products. An open cylinder can function as a two-dimensional wormhole when it joins two parallel surfaces if the physics of 90° corners is overlooked. The cylinder is a component of $\partial(I \times D^2) = (\partial I \times D^2) \cup (I \times S^1)$ with $\partial I \times S^1$ a two-part common boundary. "Flatland" inhabitants in one surface could move to the parallel surface by entering this $I \times S^1$ that bounds an empty $I \times D^2$. A bisected model made of bundles of semicircles could be used to explain this tunnel to Flatlanders. Joining these semicircles at their corresponding endpoints yields the $I \times S^1$ wormhole (Figure 5). Semicircles are used as factors because they stack together in a surface better than bundles of circles, but the Flatlanders need to know the circles in the product stack in a bundle over an interval orthogonal to their 2-space.

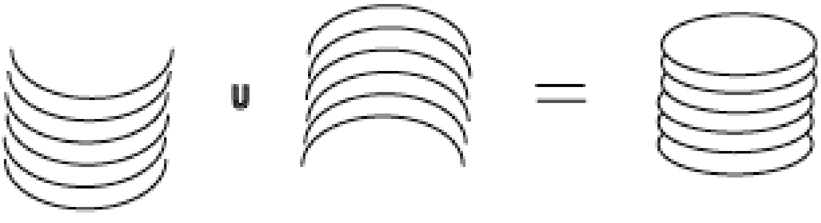


Figure 5. Cross sections of a bisected cylinder projected into a plane.

Next, this kind of bisected model is used to construct an $I \times S^2$ wormhole embedded in 4-space. As illustrated in Figure 6, a bundle of northern hemispheres (H_+^2) is joined at corresponding edges to a bundle of southern hemispheres (H_-^2) to yield the interval of spheres orthogonal to our 3-space: $(I \times H_+^2) \cup (I \times H_-^2) = I \times S^2$. This product is the open spherical hypercylinder that is the wormhole often studied as a way of connecting parallel 3-spaces. Its geometry results in a spherical metric normal to its axis. As one component of $\partial(I \times B^3)$, $I \times S^2$ has a two-part common boundary with $\partial I \times B^3$. This second component of $\partial(I \times B^3)$ would be two B^3 entrance and exit balls. The product $I \times T^2$ is a different kind of open wormhole with a toroidal cross-section (Thurston, 1998, p. 2547). Since $I \times T^2 \subset R^1 \times R^4 = R^5$ its metric is flat Euclidean in 5-space where observers in this wormhole could see repeating images of their vessel off to the sides as it moves along the wormhole axis.

The closed wormhole, $S^1 \times S^2$, is a common boundary in 5-space for $\partial(D^2 \times B^3)$. It bounds each of two different products: $\partial(S^1 \times B^3) = \partial(D^2 \times S^2) = S^1 \times S^2$. We can decompose $S^1 \times S^2$ using a model like the one used for $I \times S^2$ since these products have similar metric. Decomposition steps begin with bisecting the $S^1 \times S^2$ torus of spheres as was done for $I \times S^2$ to create two tori of hemispheres $S^1 \times H_+^2$ and $S^1 \times H_-^2$. These tori are then cut apart and straightened before they are projected into 3-space as side-by-side cylinders composed of hemispheres (Figure 6). The identification of the ends of each of these solid bundles of hemispheres and joining together corresponding hemispheres along the identity map of their meridian edges recovers $S^1 \times S^2$ as a single torus of spheres: $(S^1 \times H_+^2) \cup (S^1 \times H_-^2) = S^1 \times S^2$. The bundle of meridian edges in this identity map will recover one of the flat tori from subsets of T^2 in $S^1 \times S^2$. Also, a locus of S^2 centers is a "polar circle" and a locus of S^1 centers is a "polar sphere" in $S^1 \times S^2$.

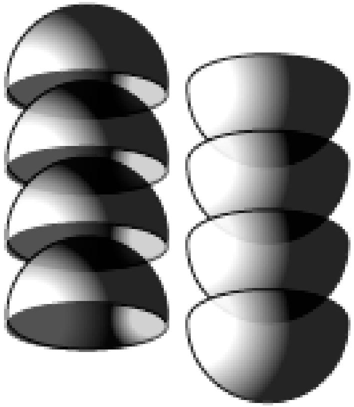


Figure 6. Matching segments of projections of hemispherical hypercylinders. Joining these segments at adjacent edges of hemispheres (and compacting) recovers the spheres in part of an open wormhole.

Another closed wormhole might be $S^1 \times T^2$ or T^3, here noted as the common boundary for $\partial(D^2 \times [D^2 \times S^1])$ and the triple cylinder. This is also the three-torus, an important Euclidian figure when embedded in 6-space: $S^1 \times T^2 \subset R^2 \times R^4 = R^6$. Geometrically this can be described as a cube with opposite faces identified and, as such, has been proposed as a model for a closed flat universe. The distortion of this geometry resulting from confinement in R^5 or R^4 is an area for further study. A fold-out decomposition of a three-torus can yield a solid cube or a solid hexagonal prism because T^3 has geometrically different "repeat axes" at 90^0 or 60^0.

4. Disk Products and the Hypersphere

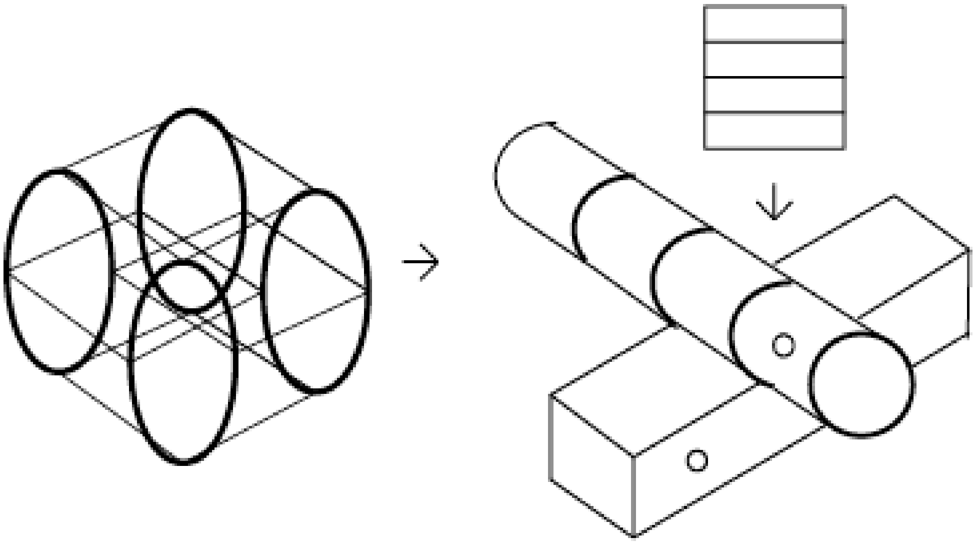


Figure 7. Fold-out of a projection of a square prism cylinder with the corresponding polyhedral torus unfolded in the upper right.

The product of a polygonal surface and a disk yields a hypersolid prism cylinder. Figure 7 illustrates the square prism cylinder $\partial(P_4^2 \times D^2)$ that is composed of disjoint interiors of $P_4^1 \times D^2$ and $P_4^2 \times S^1$ enclosed by their common boundary $P_4^1 \times S^1$. Its fold-out is marked with an "O" to illustrate coinciding surfaces when the solid figure (drawn as a projection of an outline) is recovered in R^4. The common boundary $P_4^1 \times S^1$ joining the solid rings in the square prism cylinder is shown as a flat torus fold-out with parallel creases. A summary of products and boundaries featuring prism cylinders is presented in Table 1 using familiar bounded two-dimensional factors F_1 and F_2.

Table 1. Bounded surfaces as factors for geometric products.

	Factors are *n* sided P_n^2 and *m* sided P_m^2	Both factors P_4^2 or I x I	Both factors D^2	Different factors P_n^2 and D^2
Product of factors (F_1 x F_2)	Coxeter (hypersolid) double prism P^2 x P^2	Hypersolid hypercube I x I x I x I or P_4^2 x P_4^2	Hypersolid double cylinder D^2 x D^2	Hypersolid prism cylinder P^2 x D^2
Boundary of factors	Polygons P_n^1 and P_m^1	Squares P_4^1 or ∂(I x I)	Circles S^1	Polygon and Circle
Product of one factor with boundary of other factor	Solid polyhedral tori P_n^2 x P_m^1 and P_n^1 x P_m^2	Solid polyhedral tori I x I x ∂(I x I) or P_4^2 x P_4^1 and ∂(I x I) x I x I or P_4^1 x P_4^2	Solid tori D^2 x S^1 and S^1 x D^2	*n*-faced solid torus P_n^2 x S^1 and *n*-segmented solid torus P_n^1 x D^2
Boundary of a product of factors ∂(F_1 x F_2)	Manning double prism ∂(P_n^2 x P_m^2)	Hypercube ∂(I x I x I x I)	Double cylinder ∂(D^2 x D^2)	Prism cylinder ∂(P_n^2 x D^2)
Product of boundaries of each factor ∂F_1 x ∂F_2	Polyhedral torus P_n^1 x P_m^1	Four-by-four polyhedral torus P_4^1 x P_4^1	Flat torus S^1 x S^1	*n*-faced flat torus P_n^1 x S^1

Replacing the square P_4^1 and square surface P_4^2 in the solid products comprising the square prism cylinder with the circle S^1 and the disk D^2 results in solid tori S^1 x D^2 and D^2 x S^1. Their union of disjoint interiors at common boundary T^2 yields a double cylinder and has a fold-out in which a second solid cylinder takes the place of the square prism in Figure 7. The double cylinder ∂(D^2 x D^2) bounds the aspherical product D^2 x D^2. This product of disks is topologically but not geometrically equivalent to a four-dimensional ball B^4. Products of bounded factors that are topologically a ball B^d have planar cross sections that geometrically include the diameter of one factor times the diameter of the other. These rectangular cross sections appear in I x D^2, D^2 x D^2, D^2 x B^3, B^3 x B^3, etcetera. They reduce the “roundness” of these products compared with B^d such as B^4 or B^6 whose planar cross sections are always a disk. Further study on the location of these rectangular cross sections would help characterize the shape of these objects.

The boundary of B^4 - the 3-sphere S^3 - can be regarded as a topological union of two S^1 x D^2 solid tori. Geometrically, however, the S^1 x D^2 products bound D^2 x D^2, not B^4. Since orthogonal solid tori constituting ∂(D^2 x D^2) and the 3-sphere S^3 both contain central polar circles and have similar common boundary enclosures, one might ask how the geometric lower dimensional composition of the solid tori in S^3 differs from the S^1 x D^2 and D^2 x S^1 solid torus components of ∂(D^2 x D^2).

In contrast to the double cylinder with its Euclidean metric, the 3-sphere (or n-sphere) cannot be represented as the discrete union of any small set of geometric products of lower dimensional factors. An intuitive rather than formal way of constructing S^3 by constructing its union of solid tori will make use of an “inflation” of a hypercube. Analogous to inflating a cube to approximate a sphere (Figure 8), a radius-dependent radial expansion (Δr inverse to radius r) of a hypercube from its center will approach the spherical geometry of S^3. Orthogonal cross sections of an expanded square in the equatorial belt in Figure 8 become approximate quarter circles of great circles on the sphere. Thus, orthogonal square cross sections in each of the expanding cubes comprising the inflating solid rings of the hypercube become approximate quarter spheres, similar to the polar caps in Figure 8, with their centers merging with the center of the resulting 3-sphere. This model shows that the tori constituting a 3-sphere can consist of bundles of polar caps of great spheres and thereby differ from bundles of meridian disks enclosed by T^2 that comprise the $S^1 \times D^2$ tori in a double cylinder. Check your favorite topology texts to see if they obscure these important geometric observations. Furthermore, polar caps in the solid tori of S^3 separated by a 180° rotation around the center of S^3 belong to the same great sphere.

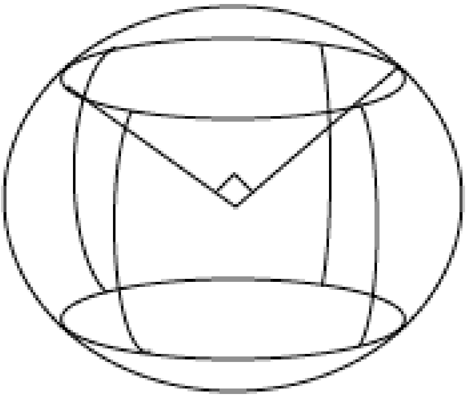


Figure 8. A sphere showing four center-facing sides that each occupy a quarter of the equatorial belt and polar caps that subtend a 90 degree angle at the center.

Geometric products have guided the construction of figures that range from simple orthotopes and prisms to difficult to visualize toroidal and spherical objects. By applying the boundary operator and focusing on the hypercube this exploration led to a collection of basic figures embedded in 4-space other than the remaining regular polytopes. A look at wormholes yielded products embedded in 5-space and 6-space as well. Fold-out decompositions aided visualizing tori in most products and bundle decompositions revealed the lower dimensional subsets within many figures. In the process of examining these figures some subtle differences between topology and geometry were illuminated.

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REFERENCES

Banchoff, Thomas F.: 1990, *Beyond the Third Dimension*, W. H. Freeman, New York, NY.

Coxeter, H. S. M.: 1974, *Regular Complex Polytopes*, Cambridge Univ. Pr., Cambridge, UK.

Kelley, John L.: 1955, *General Topology,* Springer, New York, NY.

Manning, Henry P.: 1914, *Geometry of Four Dimensions*, Macmillan, New York; reprinted in 1956 by Dover, New York, NY.

Sommerville, D. M. Y.: 1929, *An Introduction to the Geometry of N Dimensions*, Methuen & Co. Ltd, London; reprinted in 1958 by Dover, New York, NY.

Thurston, William P.: 1998, “How to see 3-manifolds”, *Class. Quantum Grav.* **15**, p. 2547.

Weeks, Jeffrey R.: 1985, *The Shape of Space*, Marcel Dekker, New York, NY.